AIAA 81-4256

Kinematic Barrier for Gravity Waves on Variable Currents

Wilson C. Chin*
Massachusetts Institute of Technology, Cambridge, Mass.

The kinematic barrier encountered by a steady linear-gravity wave propagating on a variable current in deep water, corresponding to a zero in the group velocity, is re-examined by fixing wave action flux and frequency in the weakly nonlinear limit. Computed results reveal several intriguing properties of gravity waves not previously discussed. For small values of a nondimensional action flux, large but bounded wave amplitudes not unlike those obtained in the singular linear approximation are found near the linear focus, while for large values all flow gradients are weak and smooth; in all cases, transmission through the linear focus is possible. Some apparent inconsistencies and contradictions regarding the role of nonlinearity in effecting passage through the linear focus, arising from the works of Smith and Holliday, are also discussed and clarified by delineating several contrasting assumptions underlying the two models; the explanations, in the author's opinion, tend to support the results of the present study, but more comprehensive study is needed. Calculations also show how, for "positive" action fluxes, the position of the linear focus separates single- and triple-valued solutions for action density when plotted against the propagation coordinate; the same focus, for "negative" fluxes, separates regions containing nonexistent and double-valued solutions. These results are compared with linear theory.

Introduction

THE kinematic "stopping" barrier encountered by a steady linear-gravity wave propagating through a variable current in deep water is well known to hydrodynamicists (e.g., see Phillips[1] or Gargett and Hughes[2]) and corresponds, as noted in Landahl,[3] to a zero in the local wave group velocity. The resulting ray coalescence leads to unbounded amplitude densities. The effects of weak nonlinearity and high-order dispersive modulations are therefore of significant interest (Ref. 4 provides a systematic review of recent developments in kinematic wave theory); they may be expected to remove, to some extent, the linear wave amplitude singularity and to allow passage through the focus. Several studies assuming weak nonlinearity only are already available. Crapper,[5] for example, following Whitham's[6] kinematic wave approach, conserves wave action flux and frequency using an approximate Lagrangian of Lighthill; Peregrine and Thomas[7] repeated Crapper's study using an improved Lagrangian based on some exact results of Longuet-Higgins, while an earlier study of Holliday[8] proceeded directly from the complete conservation laws. In some instances, these studies produced large-amplitude solutions qualitatively like those obtained from linear theory, thus invalidating the "slowly varying" assumption implicit in the governing low-order modulation equations. Attempts to extend these analyses by retaining in the phase equation high-order dispersive terms based on the primary "linear" harmonic appear, for example, in Smith,[9] Peregrine,[10] Stiassnie and Dagan,[11] and Smith[12]; these authors, essentially, show how the actual wave modulations now are locally described by Airy functions. These high-order results and their relationship to low-order weakly nonlinear theory and the results of the present investigation will be fully discussed later.

Note that the present paper examines only the effect of weak nonlinearity on wave propagation through the linear wave barrier. The steady nonlinear equations used in our analysis, as discussed in Whitham,[6] conserve wave action flux and frequency throughout, consistent with the approaches taken in Refs. 5,7, and 8. These latter studies considered, however, the effects of nonlinearity only in the immediate local vicinity of the linear focus and did not specifically address the resulting changes to the flow in the distant near field and far field nor the possibility of nonlinear wave reflection. Here a more complete study is pursued where the basic flow parameters span a greater range of positive and negative mean flow speeds and wave action flux. For small values of a nondimensional action flux, large but bounded wave amplitudes not unlike those obtained in the singular linear approximation are found near the linear focus, while for large values all flow gradients are weak and smooth. In all cases, passage through the linear focus was possible. Calculations also show how, for "positive" wave action fluxes, the position of the linear focus separates single- and triple-valued solutions for the wave action density coordinate when plotted against the propagation coordinate; the same focus for "negative" fluxes separates regions containing nonexistent and double-valued solutions. A brief analysis follows.

Analysis and Basic Results

The low-order kinematic equations governing the steady propagation of weakly nonlinear gravity waves through variable currents in deep water conserve frequency and wave action flux, that is, $VK+sg^{1/2}K^{1/2}+sAK^3=W_0$ and $AV+\frac{1}{2}sg^{1/2}AK^{-1/2}+3sA^2K^2/2=B_0$, respectively (e.g., see Hayes[13]). Here, K is the wavenumber, A the wave action density, g the acceleration due to gravity, V the speed of the mean flow, W_0 a prescribed frequency characterizing the wave motion, B_0 a given action flux, and $s=+1$ or $s=-1$ accordingly as the wave advances or recedes with respect to V.

When V varies with streamwise position, these equations contain both the effects of radiation stress interaction and nonlinear wave focusing. The "prescribed" V here uncouples from any wave growth or decay, however, because its dependence on the "vertical" modal coordinate has been suppressed (any nonzero change to V would unrealistically imply infinite changes to the mean flow energy since deep water is assumed). For convenience, we introduce the barred nondimensional variables $\bar{V}=4W_0V/g$, $\bar{K}^{1/2}=g^{1/2}K^{1/2}/2W_0$, $\bar{A}=64W_0^5A/g^3$, and $\bar{B}=256W_0^6B_0/g^4$. The foregoing equations reduce, then, to $VK+2sK^{1/2}+sAK^3=1$ and $AV+sAK^{-1/2}+3sA^2K^2/2=B$ where we have dropped all overbars and assumed without loss of generality that W_0 is

Received Jan. 30, 1980; revision received Aug. 7, 1980.

*Research Staff, Department of Aeronautics and Astronautics (presently, Engineering Manager, Pratt and Whitney Aircraft Group, East Hartford, Conn.). Member AIAA.

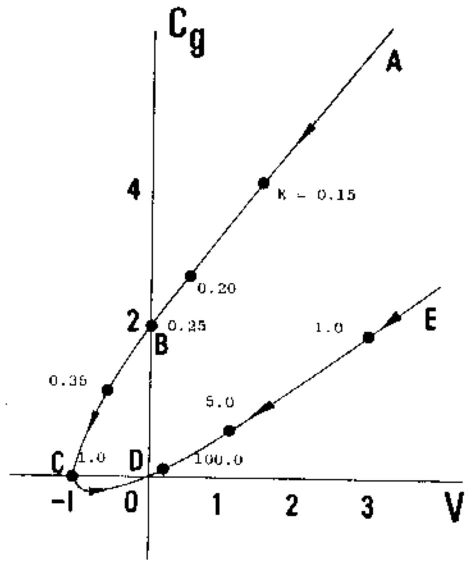

Fig. 1a Linear gravity waves (arrows indicate direction of increasing wavenumber).

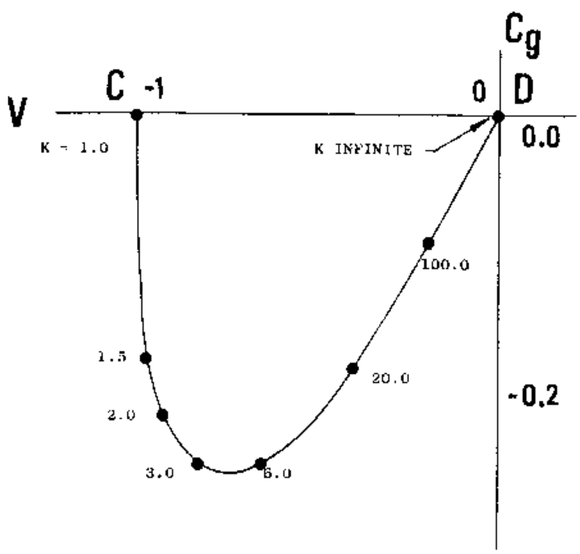

Fig. 1b Linear gravity waves (enlarged view).

positive (negative W_0 solutions are easily obtained by reflection). We examine below the complete family of nonlinear solutions obtained over an extended range of mean flow speeds and wave action fluxes and, for completeness, review first some consequences of linear theory.

The Linear Case

In this limit the nonlinear frequency equation reduces to $VK+2sK^{1/2}=1$ and a nondimensional group velocity $C_g=V+sK^{-1/2}$ can be defined (note that $A=B/C_g$). These are conveniently rearranged as $V=(1-2sK^{1/2})/K$ and $C_g=1/K-s/K^{1/2}$. A complete family of solutions is produced by varying K throughout the range of positive real values. The solutions so obtained are shown in Fig. 1, where C_g is plotted against V. Along ABCDE, ABCD and DE are derived using $s=+1$ and -1, respectively, and local wavenumber solutions are indicated. Now consider a wave approaching C from A. As the wave nears C, the group velocity C_g decreases to zero, producing singular values in A, while the wavenumber K increases to unity. In the range $-1<V<0$ the medium supports two sets of waves, one with positive C_g and the other with negative C_g; thus, the wave approaching C from A may reflect from the focus along the second branch, but it then encounters a second focus at D where it must remain trapped. For positive V, the medium supports two sets of waves again, but both have positive group velocity relative to a ground-fixed observer. A wave approaching D from E, therefore, cannot reflect and is once more trapped; however, the waves found at D have infinite wavenumber and are most likely damped by viscosity (e.g., see Fig. 1b). Wave solutions do not exist for values of V less than -1 within the framework of linear theory. The foregoing description is much more comprehensive than those given in Refs. 1,5,7, and 8 and has not been previously outlined.

The Nonlinear Case

As before, we plot all wave quantities against the nonuniformity V. Because A and K now are nonlinearly coupled, there may exist additional solutions beyond those shown in Fig. 1. These can be obtained by rewriting the governing equations in the form

$$A=\frac{-(1-sK^{1/2})\pm((1-sK^{1/2})^2+2sBK^4)^{1/2}}{sK^3}$$

$$V=\frac{1-2sK^{1/2}-sAK^3}{K}$$

As before, solutions for A, given s and B, are obtained by varying K over the range of positive real numbers; only for positive real A are the corresponding V calculated. The nonlinear solutions obtained in this manner are shown in Figs. 2a and 2b, for positive B and negative B, respectively.

Recall now a previous example where (in Fig. 1a) the wave approaches C from A. Linear theory disallows any transmission through the focus although reflection from it is possible. Figure 2a indicates bounded values in A obtained near $V=-1$, agreeing with Refs. 5, and 7, and 8. However, Fig. 2a also shows that passage through the focus into regions of very negative V, where linear theory gives no solution, is always possible. In addition, the expected rapid localized growth in wave action density, which arises out of ray coalescence, is observed only for small values of B; for large B, no evidence of any near-singular behavior is found and the transmission is completely smooth. Note that the action solution left of the linear focus, for a given B, is single-valued; however, to the right, there always exists a value of V for every prescribed B, beyond which triple-valued wave action solutions are found. Linear theory provides two solution branches with positive C_g; the added curve obtained here corresponds to a "strong" solution typified by extremely high amplitudes. The physical significance of these multivalued solutions is uncertain at the present time; the possibility of hydraulic jumps in the kinematic entities A and K through which frequency and momentum flux are conserved has been raised by Chin,[14] but this is speculative. (Reference 14 mainly dealt with hydraulic jumps in shallow-water flows that include the added presence of finite-amplitude disturbance waves which conserve total energy, total mass, total momentum, and wavenumber.) Figure 2b displays "reflected wave" solutions obtained for negative B. No solutions are available for large negative V; however, to the right two action solutions are found, one weak and one strong, beyond a critical value of V for a given B. Linear theory from Fig. 1, disallows negative C_g for $V>0$; here, the nonlinear model raises the possibility of flows allowing partial transmission and reflection. Of course, the averaging method used here assumes only one family of waves and any further comment would be only speculative; however, that possibility is suggested by the linear theory of Stiassnie and Dagan[11] who include the high-order terms and derive the required reflection and transmission coefficients. More discussion on this point follows.

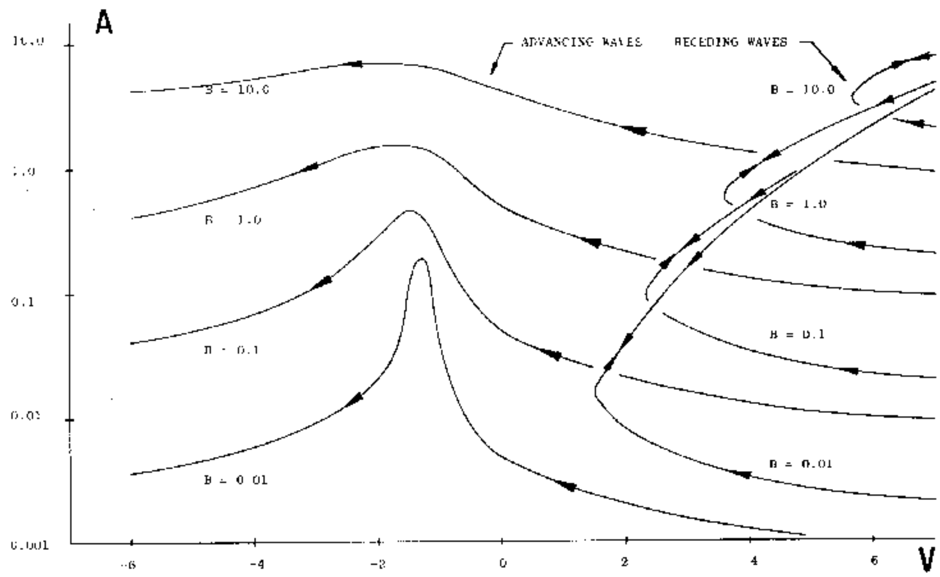

Fig. 2a Nonlinear gravity waves with positive action flux (arrows indicate direction of wavenumber growth).

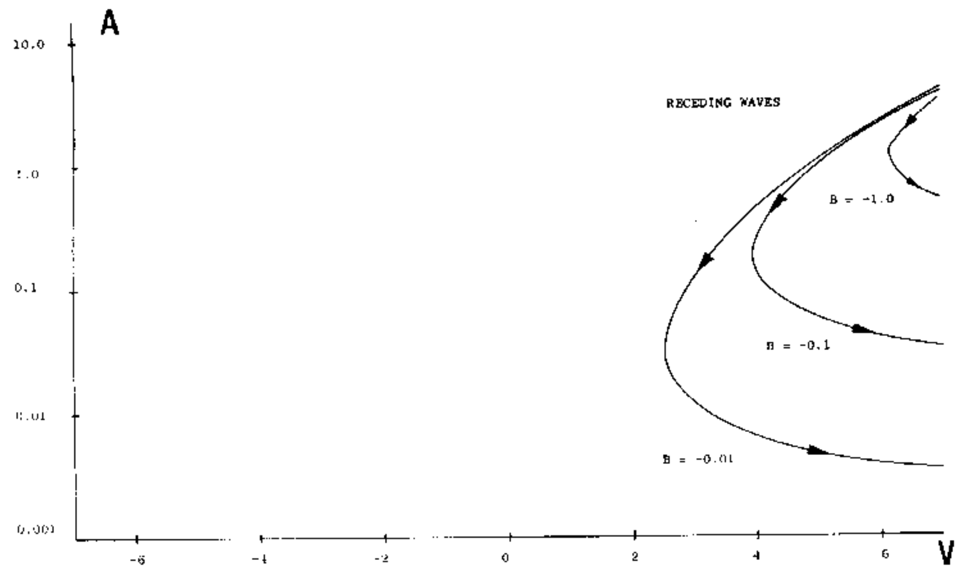

Fig. 2b Nonlinear gravity waves with negative action flux (arrows indicate direction of wavenumber growth, advancing wave solutions have unrealistic negative action densities).

Discussion and Conclusion

Our nonlinear analysis shows that there exist additional solution branches to the modulation equations for gravity wave flows beyond those predicted from simple linear theory; the approach, consistent with previous studies, examines a much broader range of mean flow velocities and wave action fluxes, providing plots for action density against mean speed that are conveniently related to the propagation coordinate. The results suggest that propagation through the linear focus is always possible and, provided the initial action fluxes are sufficiently large, all streamwise gradients in the wave action density will remain small, thus satisfying the "slowly varying" requirement implicit in the use of kinematic wave theory. For small fluxes, streamwise variations in action density are reminiscent of linear theory, although all values are bounded; the rapid gradients observed here suggest that high-order dispersive terms neglected in the low-order equations used here may be important.

A number of items appearing in the recent literature deserve clarification. The original work of Phillips[1] and Gargett and Hughes[2] first pointed out the existence of the kinematic limit; this "stopping singularity" is linear and low-order in the sense that both the Stokes nonlinearity and the high-order dispersive corrections to phase based on the primary linear harmonic are not considered. Holliday[8] extended this work by averaging an equivalent set of conservation equations, however, including the effects of weak nonlinearity. His results, supported in the present analysis, suggest that nonlinearity removes the singularity inherent in linear theory and allows passage through the focus. The subsequent work of Smith,[12] on the other hand, includes all the competing effects of flow inhomogeneity, weak nonlinearity, and high-order dispersion; the wave modulations there are governed by a nonlinear Schrodinger equation with variable coefficients reducible, in the uniform flow limit, to an equation of Davey and Stewartson.[15] Solutions for the usual complex wave amplitude are Painlevé transcendents of the second kind (qualitatively, the solutions resemble Airy functions with the transition from sinusoidal to exponential behavior occurring at a slightly displaced position). Smith finds that, essentially, the large-amplitude solutions obtained differ little from those generated using the same high-order equation but without the nonlinear term, thus contradicting Holliday's suggestion (and therefore our result) that finite-amplitude effects remove the wave barrier. "The error in Holliday's conservation equation analysis," Smith notes, "is that the mean total fluxes of mass and energy are evaluated on the hypothesis that there is no reflected wave." This observation is correct, but the differences really arise because of contrasting assumptions.

Smith, in fact, *assumes* that both a frequency and a wavenumber vector can be found which "satisfy the conditions for there to be a linear-theory reflection line," thus presupposing that a "near-singular" flow whose component of "linear group velocity" normal to the reflection line vanishes does exist; on this basis, he derives a governing nonlinear Schrodinger equation expanding all flow quantities about the conditions which define the linear focus. The assumption that such conditions can be found, strictly speaking, places limitations on the generality of the final results. For example, if we had in our analysis determined the effects of nonlinearity by expanding about linear conditions, the effects of nonlinearity would certainly remain weak, being dominated by the strong gradients of linear theory. For the advancing waves (shown in Fig. 2a) characterized by small values of B, this analytical approach is justifiable; but the same figure also shows that, without linearizing assumptions, advancing waves with large initial B while large in amplitude are characterized by weak flow gradients and the lack of any near-linear behavior. In this case, Smith's premise that a "reflection line" can be found is untrue and his comments on Holliday's results do not apply.

The comparison with Holliday's work is somewhat misleading also because Smith assumes a perfectly reflected wave (see Eq. (8) of Ref. 14) whereas Holliday's analysis implicitly assumes one that is totally transmitted (his averaging uses only the family of waves moving in the incident direction). The recent and more general work of Stiassnie and Dagan,[11] we might note, allows both reflection and transmission; these authors show how their high-order wave model provides a steady transition from Holliday's "totally transmitted" to Smith's "totally reflected" solution, allowing for the possibility of intermediate flows characterized by partial reflection and partial transmission. Unfortunately, their study did not consider at all the effects of nonlinearity; thus, the availability of any fully conclusive results must await the matching of Smith's "inner" nonlinear Schrodinger equation to, say, our "outer" nonlinear kinematic solution. The solutions displayed in Fig. 2a, again, do indicate that for sufficiently large B passage through the linear focus is possible without strong streamwise gradients and without reflection; apparently, these incident waves possess enough momentum that they can "march" through the linear focus very much unopposed. Quite possibly, these solutions also provide satisfactory solutions to Smith's high-order nonlinear Schrodinger equation. Because the linear singularity and any semblance of it simply do not exist and because all streamwise gradients are small, the effect of the high-order phase-related A_{xx} dispersion term should be negligible.

Finally, we note in passing that experimental support for the large-amplitude wave forms determined near the linear focus appears in Lewis, Lake, and Ko[16]; there, in a steady coordinate system moving with the phase velocity of the supporting nonuniform internal wave flow, large amplitudes are observed near zeroes in the linear group velocity. A study for shallow-water waves complementary to the present investigation, connecting wave focusing to hydraulic jump formation, appears in Ref. 17, motivated by the wave model presented earlier in Ref. 14.

Acknowledgments

I wish to thank M. Landahl for many helpful discussions, and S. Widnall, L. Howard, and D. Benney for their constructive criticism. This research was supported by Air Force Office of Scientific Research under Contract AFOSR-74-2730.

References

[1] Phillips, O.M., *The Dynamics of the Upper Ocean*, Cambridge University Press, England, 1969.

[2] Gargett, A.E., and Hughes, B.A., "On the Interaction of Surface and Internal Waves," *Journal of Fluid Mechanics*, Vol. 52, 1972, pp. 179-191.

[3] Landahl, M.T., "Wave Mechanics of Breakdown," *Journal of Fluid Mechanics*, Vol. 56, 1972, pp. 775-802.

[4] Chin, W.C., "Effect of Dissipation and Dispersion on Slowly Varying Wavetrains," *AIAA Journal*, Vol. 18, Feb. 1980, pp. 149-158.

[5] Crapper, G.D., "Nonlinear Gravity Waves on Steady Nonuniform Currents," *Journal of Fluid Mechanics*, Vol. 52, 1972, pp. 713-724.

[6] Whitham, G.B., *Linear and Nonlinear Waves*, John Wiley and Sons, N.Y., 1974.

[7] Peregrine, D.H. and Thomas, G.P., "Finite Amplitude Waves on Nonuniform Currents," Paper presented at IUTAM Symposium on Surface Gravity Waves on Water of Varying Depth, Canberra, Australia, July 1976.

[8] Holliday, D., "Nonlinear Gravity-Capillary Surface Waves in a Slowly Varying Current," *Journal of Fluid Mechanics*, Vol. 57, 1973, pp. 797-802.

[9] Smith, R.W., "The Reflection of Short Gravity Waves on a Nonuniform Current," *Mathematics Proceedings*, Cambridge Philosophy Society, Vol. 78, 1975, pp. 517-525.

[10] Peregrine, D.H., "Interaction of Water Waves and Currents," *Advances in Applied Mechanics*, Vol. 16, 1976, pp. 9-117.

[11] Stiassnie, M. and Dagan, G., "Partial Reflection of Water Waves by Nonuniform Adverse Currents," *Journal of Fluid Mechanics*, Vol. 92, 1979, pp. 119-130.

[12] Smith, R., "Giant Waves," *Journal of Fluid Mechanics*, Vol. 77, 1976, pp. 417-431.

[13] Hayes, W.D., "Group Velocity and Nonlinear Dispersive Wave Propagation," *Proceedings of the Royal Society*, London, Series A332, 1973, pp. 199-221.

[14] Chin, W.C., "Kinematic Wave Approach to Hydraulic Jumps with Waves," *Journal of Hydronautics*, Vol. 13, Oct. 1979, pp. 120-124.

[15] Davey, A., and Stewartson, K., "On Three-Dimensional Packets of Surface Waves," *Proceedings of the Royal Society*, London, Series A338, 1974, pp. 101-110.

[16] Lewis, J.E., Lake, B.M., and Ko, D.R.S., "On the Interaction of Internal Waves and Surface Gravity Waves," *Journal of Fluid Mechanics*, Vol. 63, 1974, pp. 773-800.

[17] Chin, W.C., "Wave Focusing and Hydraulic Jump Formation," *Journal of Hydronautics*, Vol. 14, July 1980, pp. 94-96.